TURBULENT MIXING OF FLUIDS WITH DIFFERENT DENSITIES FLOWING THROUGH A PIPE

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A one-dimensional model of a disperse mixture in a turbulent stream is constructed, with the mutual effect of mixture concentration and turbulence intensity taken into account.

We consider the mixing of mutually soluble fluids with different densities when one flows behind the other or when part of one fluid spreads in a stream of the other in a circular pipe.

In a stream with a fully developed turbulence, where the mixing rate is high, a separation of the components from the mixture in a gravity field will be only slight. For this reason, the average flow through a circular pipe will be assumed axially symmetric, regardless of whether the pipe is horizontal or inclined. The mixing of fluids will be considered at the instant of time when the concentration has become almost uniform across a pipe section, as a result of radial diffusion, and the deviations from the mean-over-the-section concentration are only slight. These deviations are due to a convective transfer of the solute, as a consequence of a nonuniform velocity profile. At instants of time determined by the diffusivity $t_* = d^2/D_0$ (d denoting the pipe diameter, D_0 denoting the characteristic value of radial turbulent diffusivity), the mixing zone is far wider than the pipe diameter and the average flow can be considered almost linear and parallel to the pipe axis.

We introduce a cylindrical system of coordinates whose axis coincides with the pipe axis in the direction of flow. In accordance with the conventional flow model, the system of equations describing a turbulent flow of an inhomogeneous fluid is

$$\frac{1}{\rho_0}\cdot\frac{\partial}{\partial x}p=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\nu_t\frac{\partial U}{\partial r}\right)-g\sin\omega,\qquad \frac{\partial}{\partial r}p=0,$$
$$\frac{\partial rU}{\partial x}+\frac{\partial rV}{\partial r}=0, \tag{1}$$
$$\frac{\partial C}{\partial t}+\frac{\partial CU}{\partial x}+\frac{\partial CV}{\partial r}=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(rD_t\frac{\partial C}{\partial r}\right)+\frac{\partial}{\partial x}\left(D_t\frac{\partial C}{\partial x}\right).$$

In deriving the first equation here, we have replaced the average density by the mean-over-the-section density, on the assumption that the radial density and concentration nonuniformities are small.

We assume a constant flow rate and, accordingly, the continuity equation in (1) will yield a constant mean velocity.

As was done in [2], we will derive an equation which describes the distribution of the mean solute concentration. This equation is

$$\frac{\partial c_0}{\partial t}=\frac{\partial}{\partial X}\left(K\frac{\partial c_0}{\partial X}\right),\quad X=x-U_0t. \tag{2}$$

The effective turbulent diffusivity K is calculated by the following formula [3]:

$$K=2a^2\int_0^1\left[\int_z^1\eta\,(U-U_0)\,d\eta\right]^2\frac{dz}{zD_t},\qquad z=\frac{r}{a}. \tag{3}$$

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From the equation of motion in system (1) we derive

$$\tau_0 \frac{r}{a} = -\rho_0 \nu_t \frac{\partial U}{\partial r}. \tag{4}$$

The rates of mass and momentum transfer in a turbulent flow are almost equal and, therefore, the coefficients ν_t and D_t, which characterize the respective transfer rates in the mainstream, can be assumed equal. With this stipulation and with equality (4), expression (3) becomes

$$K = -\frac{2a}{u_*^2} \int_0^1 \frac{\partial U}{\partial z} \left[\int_z^1 \eta (U - U_0)\, d\eta \right]^2 \frac{dz}{z},$$
$$u_* = (\tau_0/\rho_0)^{1/2}. \tag{5}$$

Expression (5) contains only one unknown quantity, namely the axial component of the average velocity, which is needed for calculating the coefficient K.

In [2, 4-8], where the mixing of mutually soluble fluids has been analyzed theoretically, the effective diffusivity was calculated on the basis of a velocity profile hypothetically the same as in a homogeneous stream. The solute was, furthermore, assumed "passive," i.e., the presence of one fluid in the stream of another was assumed to have no effect on the turbulent mass and momentum transfer processes. On the basis of such an assumption, one arrives at a homogeneous model of a dispersion with constant diffusivity.

It is entirely valid to assume a "passive" solute when the densities of both fluids are equal. If the densities are not equal but different, however, then within the mixing zone the flow parameters will be different than in the homogeneous regions. For instance, the turbulence intensity in the mixing zone is not the same as in the homogeneous regions and depends on the density as well as the concentration distribution. Moreover, this dependence is reciprocal: the turbulence intensity, in turn, determines the mixing characteristics and, therefore, the distribution of substance in an inhomogeneous stream. A similar relation has been established in [1] for a stream carrying solid particles near a wall.

Turbulent mixing at the boundary between liquids or gases has been analyzed in [9] and the following hypothesis was proposed: the energy dissipated by turbulent mixing of fluids in an inhomogeneous stream is equal to the energy dissipated by turbulence. This hypothesis will be used here for deriving the equation of balance of turbulence energy in the mainstream in a pipe, on the basis of which the transfer coefficients will then be determined and the velocity profile in the mixing zone will be found. Coefficient K can be calculated from this velocity profile; at the same time, the dispersion model obtained in this manner will be more general than the Taylor model with the effective coefficient depending on the concentration profile.

The equation of balance of turbulence energy in the homogeneous mainstream in a pipe, except for the region where appreciable energy transfer by diffusion occurs near the axis, is on the basis of measurements made by Laufer [10]:

$$\frac{\rho b^{3/2}}{\gamma^4 l} = \rho \nu_t \left(\frac{\partial U}{\partial r} \right)^2. \tag{6}$$

In this equation the left-hand side represents dissipation of turbulence energy, the right-hand side represents dissipation of flow energy due to turbulence friction. For a flow of an inhomogeneous fluid, according to the hypothesis proposed in [9], to the right-hand side of Eq. (6) must be added another term which will account for the dissipation of flow energy due to turbulent mixing. The dissipation of flow energy due to turbulent mixing alone is [9]

$$\rho D_t \frac{\partial C}{\partial x_\alpha} \left(\frac{\partial \mu}{\partial p} \right)_C \frac{\partial}{\partial x_\alpha} p, \quad \alpha = 1, 2, 3. \tag{7}$$

Well known thermodynamic relations yield the following chain of equalities:

$$\left(\frac{\partial \mu}{\partial p} \right)_C = \left(\frac{\partial}{\partial C} \cdot \frac{1}{\rho} \right)_p = -\frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial C} \right)_p.$$

Quantity (7) expressed in terms of both this equality and formula

$$C = \frac{d_1 (\rho - d_2)}{\rho (d_1 - d_2)},$$

will be added on the right-hand side of Eq. (6). As a result, we have an equation of balance of turbulence energy in the mainstream of an inhomogeneous fluid through a pipe:

$$\frac{b^{3/2}}{\gamma^4 l}=\nu_t\left(\frac{\partial U}{\partial r}\right)^2-\frac{\sigma D_t}{d_1}\cdot\frac{\partial p}{\partial x}\cdot\frac{\partial C}{\partial x},\ \sigma=\frac{d_1-d_2}{d_1}. \tag{8}$$

The transfer coefficients ν_t and D_t in the Kolmogorov hypothesis, like the dissipation of turbulence energy, are defined in terms of the turbulence intensity b and a linear scale factor l. Dimensional analysis yields

$$\nu_t=l\sqrt{b},\ D_t=l\sqrt{b}. \tag{9}$$

With the aid of these equalities and the balance equation (8), we determine the turbulent viscosity as follows:

$$\nu_t=-\gamma^2 l^2\frac{\partial U}{\partial r}\left[1-\frac{\sigma}{d_1}\cdot\frac{\partial c_0}{\partial x}\cdot\frac{\partial p}{\partial x}\left(\frac{\partial U}{\partial r}\right)^{-2}\right]^{\frac{1}{2}}. \tag{10}$$

Here the average concentration has been replaced by the mean concentration, because at the given instants of time the concentration is nearly uniform along the radius.

Knowing the turbulent viscosity, one can determine the velocity profile in the mixing zone. In order to do this, we use the two-layer model of a stream through a pipe: a laminar layer at the inside surface of the pipe and a turbulent mainstream. The velocity profile in the laminar layer will be assumed linear and its thickness $y_+ = 11.5\nu/u_*$. In the turbulent mainstream the frictional stress will be assumed constant and equal to τ_0. All further estimates will be based on this stream model, which, on account of the quasi-equality $\tau = \tau_0$, applies more to the boundary-layer velocity than to the overall velocity profile in the pipe. Nevertheless, we will assume that, as in the case of a homogeneous stream, the velocity profile obtained with all those stipulations approximates the velocity distribution throughout the entire flow region till close to the pipe axis. In accordance with the stipulated stream model, the equations for determining the velocity can be written as

$$U=\frac{u_*^2}{\nu}y,\ y<y_+,\ u_*^2=\nu_t\frac{\partial U}{\partial y},\ \ y\geqslant y_+. \tag{11}$$

With the aid of Eq. (10), the second equation in (11) can be transformed into

$$u_*=\gamma l\frac{\partial U}{\partial y}\left[1-\frac{\sigma}{d_1}\cdot\frac{\partial c_0}{\partial x}\cdot\frac{\partial p}{\partial x}\left(\frac{\partial U}{\partial y}\right)^{-2}\right]^{\frac{1}{4}}. \tag{12}$$

The γl-group will be defined by the Karman formula

$$\gamma l=-\varkappa\Psi/\Psi',\ \ \Psi=\frac{\partial U}{\partial y}. \tag{13}$$

For the given flow model, the turbulence intensity in the mixing zone is not much different from that in the homogeneous stream regions. For the boundary layer, consequently, where the frictional stress is constant, the magnitude of the second term inside the brackets in (12) is much smaller than unity:

$$\left|\frac{\sigma}{d_1}\cdot\frac{\partial c_0}{\partial x}\cdot\frac{\partial p}{\partial x}\Psi^{-2}\right|\ll 1. \tag{14}$$

With this estimate and with the Karman formula, we rewrite expression (12) as follows:

$$u_*=-\varkappa(\Psi^2/\Psi')(1-\beta\Psi^{-2}),\ \ \beta=\frac{\sigma}{4d_1}\cdot\frac{\partial c_0}{\partial x}\cdot\frac{\partial p}{\partial x}. \tag{15}$$

Integrating this expression and using the estimate (14), we obtain

$$y=y_+\left[\exp\frac{\varkappa}{u_*}(U-U_+)-\frac{1}{6}\beta\Psi_+^{-2}\exp\frac{3\varkappa}{u_*}(U-U_+)\right]+\frac{1}{6}\beta\Psi_+^{-2},\ \ U_+=11.5u_*,\ \ \Psi_+=u_*^2/11.5\varkappa\nu. \tag{16}$$

The constants in this integration have been chosen so as to yield at $\beta = 0$ the universal law of velocity distribution for smooth pipes. With some error incurred by retaining only the first-power terms of the small quantity $\beta\Psi_+^{-2}$, the velocity profile is

$$\varphi = \frac{1}{\alpha}\ln\frac{1-z}{\zeta} + \frac{1}{6}\beta\Psi_+^{-2}\frac{1-z^2}{\alpha\zeta^2} + \varphi_+ - \frac{1}{6}\cdot\frac{\beta}{\alpha\zeta}\Psi_+^{-2},$$
$$\varphi = \frac{U}{U_0},\quad \alpha = \frac{\varkappa U_0}{u_*},\quad \zeta = y_+/a,\quad z = r/a. \tag{17}$$

The velocity profile in the mixing zone (17) differs from the universal logarithmic profile by the additional terms which account for the variation in flow intensity in this zone.

The pressure gradient found from the equation of motion (1) as well as the expression for Ψ_+ in (16) are used for calculating the dimensionless parameter $(1/6)\beta\Psi_+^{-2}$. After that, the velocity profile thus found is inserted into expression (5) for calculating the effective turbulent diffusivity. Performing the integration, we obtain

$$K_* = \Gamma_1 - \frac{d_1}{d_2}\left(\frac{1}{2}\lambda\sigma + \mathrm{GrRe}^{-2}\sin\omega\right)\Gamma_2\frac{\partial c_0}{\partial\xi},\quad K_* = \frac{K}{2aU_0},$$
$$\Gamma_1 = 15.6\varepsilon\left\{\frac{9}{16}\left[\ln\frac{\varepsilon\,\mathrm{Re}}{23} - \delta + \frac{1}{2} - \frac{1}{2}\left(2 - \frac{23}{\varepsilon\mathrm{Re}}\exp\delta\right)^2\right]\right.$$
$$\left. + \left(\frac{23}{\varepsilon\,\mathrm{Re}}\right)^2\left[\delta - 2\exp\delta\,(1 - \exp\delta) + \frac{1}{2}\exp 2\delta\right]\right\},$$
$$\Gamma_2 = 880\varepsilon\,(\lambda\,\mathrm{Re})^{-2}\left\{\left[\frac{3}{4}\exp\delta\left(1 - \frac{23}{\varepsilon\,\mathrm{Re}}\exp\delta\right)\right]^2\right.$$
$$+ \frac{5}{8}\left(\frac{\varepsilon\,\mathrm{Re}}{23}\right)^2\Gamma_1 + \frac{3}{8}\left(\frac{\varepsilon\,\mathrm{Re}}{23}\right)^2\left(1 - \frac{23}{\varepsilon\,\mathrm{Re}}\exp\delta\right)^3 \tag{18}$$
$$\left.\times\left[1 - \frac{3}{4}\left(1 - \frac{23}{\varepsilon\,\mathrm{Re}}\exp\delta\right)\right] + \left(\frac{23}{\varepsilon\,\mathrm{Re}}\right)^3\exp 3\delta\left(\frac{2}{9} - \frac{1}{6}\exp\delta\right)\right\},$$
$$\varepsilon = (\lambda/8)^{\frac{1}{2}},\quad \delta = \frac{0.4\,(1 - 11.5\varepsilon)}{\varepsilon},\quad \mathrm{Gr} = \frac{g\sigma d^3}{\nu^2}.$$

In deriving expression (18), we have retained the linear terms in $\beta\Psi_+^{-2}$. The Γ_1-terms are close in magnitude to the coefficient K_* determined from the Taylor formula.

The effective diffusivity calculated by formula (18) is now inserted into the equation of one-dimensional diffusion (2). The result is

$$\frac{\partial c_0}{\partial\tau} = \frac{\partial}{\partial\xi}\left[\left(\Gamma_1 + \Lambda\frac{\partial c_0}{\partial\xi}\right)\frac{\partial c_0}{\partial\xi}\right],$$
$$\Lambda = -\frac{d_2}{d_1}\left(\frac{1}{2}\lambda\sigma + \mathrm{GrRe}^{-2}\sin\omega\right)\Gamma_2. \tag{19}$$

Unlike the Taylor equation, which is linear, this equation is only quasilinear. The value of effective diffusivity depends here on the concentration distribution. If the densities of the mixing fluids are assumed equal, then Eq. (19) will yield the Taylor equation. Thus, accounting for a density difference results in a dispersion model where the effective diffusivity is a function not only of the Reynolds number but also of the concentration gradient.

We will show a method of solving Eq. (19) for the case where one fluid displaces another. This problem has the following limit conditions:

$$c_0(0,\ \xi) = 0,\quad c_0(\tau,\ -\infty) = 1,\quad c_0(\tau,\ +\infty) = 0. \tag{20}$$

In Eq. (19) we change to new variables τ and $z = \xi/(2\Gamma_1\tau)^{1/2}$. The solution to the equation in these variables is sought in the form of a series:

$$c_0(\tau,\ z) = f_0(z) + \sum_{s=1}^{\infty}\tau^{-\frac{s}{2}}f_s(z). \tag{21}$$

Functions $f_0(z)$, $f_1(z)$, ..., which are the coefficients of this series, satisfy the following ordinary differential equations:

$$f_0'' + zf_0' = 0,$$
$$f_{n+1}'' + zf_{n+1}' + (n+1)f_{n+1} = -\sqrt{2}\,\frac{\Lambda}{\Gamma_1}\sum_{s=0}^{n}f_s'f_{n-s}''. \tag{22}$$

The limit conditions for determining function $f_0(z)$ are

$$f_0(-\infty) = 1, \quad f_0(+\infty) = 0. \tag{23}$$

Inasmuch as functions $f_{n+1}(z)$ $(n = 0, 1, 2, \ldots)$ are odd, $f_{n+1}(0) = 0$. It is not difficult to show that at point $z = 0$ the derivative of a function $f_{n+1}(z)$ becomes zero. Therefore, we solve the Cauchy problem with zero limit conditions

$$f_{n+1}(0) = f'_{n+1}(0) = 0, \quad n = 0, 1, 2, \ldots \tag{24}$$

for determining the functions $f_{n+1}(z)$. The solution $f_0(z)$ is identical to the Taylor solution, namely

$$f_0(z) = \frac{1}{2}\left(1 - \sqrt{\frac{2}{\pi}}\int_0^z \exp\left(-\frac{x^2}{2}\right)dx\right). \tag{25}$$

The solution to the subsequent nonhomogeneous equations for functions $f_{n+1}(z)$ can be found by the application of well known rules with the aid of fundamental solutions to homogeneous equations. The fundamental solutions are

$$y_1(z) = \frac{d^n}{dz^n}\exp\left(-\frac{z^2}{2}\right), \quad y_2 = \frac{d^n}{dz^n}\exp\left(-\frac{z^2}{2}\right)\int_0^z \exp\frac{x^2}{2}\,dx. \tag{26}$$

For example, the solution for function $f_1(z)$ is

$$f_1(z) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\Lambda}{\Gamma_1}\exp\left(-\frac{z^2}{2}\right)\int_0^z \operatorname{sh}\frac{x^2}{2}\,dx. \tag{27}$$

NOTATION

ρ_0	is the mean-over-the-section density;
p	is the pressure;
ν_t	is the turbulent viscosity;
U	is the average longitudinal velocity;
g	is the acceleration of gravity;
ω	is the angle of pipe inclination from the horizontal;
x, r	are the cylindrical coordinates;
t	is the time;
V	is the average radial velocity;
C	is the average concentration;
D_t	is the turbulent diffusivity;
c_0	is the mean-over-the-section concentration;
K	is the effective turbulent diffusivity;
U_0	is the mean flow velocity;
X	is the distance, in the moving system of coordinates;
a	is the pipe radius;
τ_0	is the frictional stress at the inside surface of the pipe;
u_*	is the transient turbulent velocity;
b	is the turbulence intensity;
l	is the linear scale factor;
μ	is the chemical potential of mixture;
ρ	is the density of mixture;
d_1, d_2	are the densities of homogeneous fluids;
y_+	is the thickness of laminar layer;
y	is the distance from the inside pipe surface;
Ψ_+	is the derivative of velocity at the layer boundary on the turbulent side;
λ	is the hydraulic drag;
Gr	is the Grashof number;
Re	is the Reynolds number;
Γ_1, Γ_2, Λ	are the coefficients in the equation for K_*;
K_*	is the dimensionless effective diffusivity;
$\tau = U_0t/2a$	is the dimensionless time;
$\xi = X/2a$	is the dimensionless distance.

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